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Asymptotic Behavior of Solutions of $(ry^{(m)})^{(k)} \pm qy = 0$*

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For the second-order equation $y'' + qy = 0$ with q positive on $[a, \infty)$, we have, with certain restrictions on q, the WKB or Liouville asymptotic expressions for $x \to \infty$

$$y(x) \sim q(x)^{-1/4} \exp\left[\pm i \int_a^x q(t)^{1/2}\, dt\right]$$

and

$$y'(x) \sim \pm iq(x)^{1/4} \exp\left[\pm i \int_a^x q(t)^{1/2}\, dt\right]$$

for some pair of fundamental solutions. These asymptotic solutions have wide application in physics, notably in quantum mechanics. For the general second-order equation $(ry')' \pm qy = 0$, asymptotic solutions are also well known ([*1*], p. 120). In this paper we derive asymptotic solutions for higher-order equations which reduce to the known expansions in the second-order case.

Throughout we suppose that n is a positive integer $\geqslant 2$, m and k are positive integers with sum n and r and q denote positive-valued, twice continuously differentiable functions on a ray $[a, \infty)$. We consider the nth-order equation

$$(ry^{(m)})^{(k)} - (-1)^l qy = 0 \tag{1}$$

for $l = 1, 2$. We denote by $\mathbf{L}(t_0, \infty)$ the Banach space of all complex valued functions which are Lebesgue integrable on $[t_0, \infty)$.

It is convenient to consider a vector-matrix formulation of Eq. (1). We make this formulation by defining the $n \times n$ matrix $A = \{a_{ij}\}_1^n$ as follows:

$$a_{ij}(x) = \begin{cases} 1 & \text{if} \quad i \neq m \quad \text{and} \quad j = i + 1, \\ r(x)^{-1} & \text{if} \quad i = m \quad \text{and} \quad j = i + 1, \\ (-1)^l\, q(x) & \text{if} \quad i = n \quad \text{and} \quad j = 1, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

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If y is a solution of Eq. (1) and $\xi = \{\xi_i\}_1^n$ is defined by

$$\xi_i = \begin{cases} y^{(i-1)} & \text{if} \quad i = 1,\dots, m, \\ (ry^{(m)})^{(i-m-1)} & \text{if} \quad i = m+1,\dots, n, \end{cases}$$

then $\xi' = A\xi$. Conversely, if $\xi' = A\xi$, then the first component of ξ is a solution of (1).

If we make a linear transformation of the matrix equation

$$Y' = AY \tag{2}$$

by letting $Z = QY$ for some continuously differentiable nonsingular matrix Q, then a calculation shows that

$$Z' = [Q'Q^{-1} + QAQ^{-1}]\, Z. \tag{3}$$

We now choose Q to be a "shearing" transformation, i.e., a diagonal matrix. Put

$$Q = \text{diagonal}[q^{\alpha_1}r^{\beta_1},\dots, q^{\alpha_n}r^{\beta_n}]$$

where $\alpha_i = (n - 2i + 1)/2n$ and

$$\beta_i = \begin{cases} (n + 2i - 2m - 1)/2n & \text{if} \quad i = 1,\dots, m, \\ (-n + 2i - 2m - 1)/2n & \text{if} \quad i = m+1,\dots, n. \end{cases}$$

We then have

$$Q'Q^{-1} = (q'/q)\, D_1 + (r'/r)\, D_2$$

where $D_1 = \text{diagonal}[\alpha_1 ,\dots, \alpha_n]$ and $D_2 = \text{diagonal}[\beta_1 ,\dots, \beta_n]$. The observation that

$$1 + \alpha_n - \alpha_1 = \alpha_1 - \alpha_2 = \cdots = \alpha_{n-1} - \alpha_n = 1/n$$

and

$$\begin{aligned} \beta_1 - \beta_2 = \cdots = \beta_{m-1} - \beta_m = \beta_m - \beta_{m+1} - 1 = \beta_{m+1} - \beta_{m+2} \\ = \cdots = \beta_{n-1} - \beta_n = \beta_n - \beta_1 = -1/n \end{aligned}$$

yields that $QAQ^{-1} = (q/r)^{1/n}K$ where $K = \{k_{ij}\}_1^n$ is the $n \times n$ matrix

$$k_{ij} = \begin{cases} 1 & \text{if} \quad j = i + 1, \\ (-1)^l & \text{if} \quad i = n \quad \text{and} \quad j = 1, \\ 0 & \quad\text{otherwise.} \end{cases}$$

Thus Eq. (3) reduces to

$$Z' = [(q/r)^{1/n}K + (q'/q)\, D_1 + (r'/r)\, D_2]\, Z. \tag{4}$$

The characteristic polynomial of K is $p(\lambda) = (-\lambda)^n - (-1)^{l+n}$. We denote the roots of p by $\lambda_1, \ldots, \lambda_n$ where

$$\lambda_\tau = \begin{cases} \exp[i\Pi(2\tau - 1)/n] & \text{if } l = 1, \\ \exp[2\Pi i(\tau - 1)/n] & \text{if } l = 2 \end{cases}$$

and $i = \sqrt{-1}$. We put $L = \{l_{ij}\}_1^n$ where $l_{ij} = \lambda_j^{i-1}$.

THEOREM 1. *Suppose that* $(q/r)^{1/n} \notin \mathbf{L}(a, \infty)$ *and that each of* $[(q/r)^{-1/n}r'/r]'$, $[(q/r)^{-1/n}q'/q]'$, $[(q/r)^{-1/n}(r'/r)^2]$ *and* $[(q/r)^{-1/n}(q'/q)^2] \in \mathbf{L}(a, \infty)$. *Then there is a fundamental matrix Y for Eq.* (2) *such that as* $t \to \infty$, $Q(t)\, Y(t)\, E(t) \to L$ *where* $E(t)$ *is the diagonal matrix*

$$E(t) = \text{diagonal}\left[\exp\left[-\lambda_1 \int_a^t (q/r)^{1/n}\right], \ldots, \exp\left[-\lambda_n \int_a^t (q/r)^{1/n}\right]\right].$$

Proof. Let $h(t) = \int_a^t (q/r)^{1/n}$ and denote by g the inverse of h, i.e., $g(h(t)) = t$ for all $t \geqslant a$. Since $(q/r)^{1/n} \notin \mathbf{L}(a, \infty)$, we have $h(t) \to \infty$ as $t \to \infty$. If we make the change of variable $W(s) = Z(g(s))$ where Z is as in Eq. (4), then differentiation of W and a substitution using equation (4) yields

$$W'(s) = [K + \alpha(s)\, D_1 + \beta(s)\, D_2]\, W(s) \tag{5}$$

where

$$\alpha(s) = [(q/r)^{-1/n}q'/q](g(s))$$

and

$$\beta(s) = [(q/r)^{-1/n}r'/r](g(s)).$$

We now proceed to apply Theorem 8.1, of [*2*, Chap. 3].

The change of variable $s = h(t)$ proves that

$$\int_0^\infty |\alpha'(s)|\, ds = \int_a^\infty |[(q/r)^{-1/n}\, q'/q]'|(t)\, dt < \infty$$

and that

$$\int_0^\infty \alpha(s)^2\, ds = \int_a^\infty [(q/r)^{-1/n}\, (q'/q)^2](t)\, dt < \infty.$$

Since $\alpha' \in \mathbf{L}(0, \infty)$, it follows that α has a finite limit at ∞ and $\alpha^2 \in \mathbf{L}(0, \infty)$ ensures that this limit is zero. In a similar manner we have the above conclusions for β.

Next consider the characteristic values of $K + \alpha(s)\, D_1 + \beta(s)\, D_2$. We denote these by $\lambda_\tau + \gamma_\tau(s)$ for $\tau = 1, \ldots, n$ where the γ_τ are chosen so that $\gamma_\tau(s) \to 0$ as $s \to \infty$. In the notation of Theorem 8.1 of [*2*] we define for a given τ, the integer j to be in class I_1 or I_2 according as to whether the real part of $\lambda_\tau - \lambda_j$ is positive or not. The hypothesis of Theorem 8.2 is clearly satisfied if $\gamma_j \in \mathbf{L}(0, \infty)$ for $j = 1, \ldots, n$.

From the definition of γ_j it follows that

$$0 = \det[(K + \alpha(s) D_1 + \beta(s) D_2) - (\lambda_j + \gamma_j(s)) I]$$

$$= \prod_{i=1}^{n} [(\alpha_i\alpha(s) + \beta_i\beta(s)) - (\lambda_j + \gamma_j(s))] - (-1)^{l+n}.$$

Further simplification yields

$$(-1)^l = \prod_{i=1}^{n} [(\lambda_j + \gamma_j(s)) - (\alpha_i\beta(s) + \beta_i\beta(s))]\,. \tag{6}$$

Expanding the right-hand side of (6), we obtain

$$\begin{aligned}(-1)^l = (\lambda_j + \gamma_j(s))^n - (\lambda_j + \gamma_j(s))^{n-1} \sum_{i=1}^{n} (\alpha_i\alpha(s) + \beta_i\beta(s))\\ + \cdots + (-1)^n \prod_{i=1}^{n} (\alpha_i\alpha(s) + \beta_i\beta(s)).\end{aligned} \tag{7}$$

An additional calculation yields

$$0 = \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i\,. \tag{8}$$

From $\alpha(s) \to 0$, $\beta(s) \to 0$ and $\gamma(s) \to 0$ as $s \to \infty$ and Eq. (8), we now conclude from Eq. (7) that there is a number $M > 0$ such that for all sufficiently large s,

$$|(\lambda_j + \gamma_j(s))^n - (-1)^l\,| \leqslant M(|\,\alpha(s)| + |\,\beta(s)|)^2. \tag{9}$$

However, since $\lambda_j^n - (-1)^l = 0$, it follows that

$$(\lambda_j + \gamma_j(s))^n - (-1)^l = \left[\sum_{i=1}^{n} \binom{n}{i} \lambda_j^{n-i}\gamma_j(s)^{i-1}\right] \gamma_j(s),$$

and since $\gamma_j(s) \to 0$ as $s \to \infty$, we conclude from Eq. (9) that there is a number $N > 0$ such that for all sufficiently large s,

$$|\,\gamma_j(s)| \leqslant N(|\,\alpha(s)| + |\,\beta(s)|)^2.$$

Therefore $\gamma_j \in \mathrm{L}(0, \infty)$ and Theorem 8.1 of [2] is applicable.

We need now to find the characteristic vectors of K. However if $\eta = \{\lambda_r^{i-1}\}_{i=1}^n$, then it is easily verifiable that $K\eta = \lambda_r\eta$. Hence there is a solution φ_τ of $\varphi_\tau' = [K + \alpha D_1 + \beta D_2]\,\varphi_\tau$ and a number $s_0 \geqslant 0$ such that as $s \to \infty$,

$$\varphi_\tau(s) \exp\left[-\int_{s_0}^{s} [\lambda_\tau + \gamma_\tau(u)]\,du\right] \to \eta.$$

However, since $\gamma_\tau \in \mathbf{L}(0, \infty)$, it is sufficient to take φ_τ such that $\varphi_\tau(s)\,e^{-\lambda_\tau s} \to \eta$ as $s \to \infty$. Thus if we let W be the matrix solution of (5) whose columns are $\varphi_1, \ldots, \varphi_n$ and define Y by $Y(t) = Q(t)^{-1}W(h(t))$, the theorem is proved.

If we denote the first element in the τth column of Y by y_τ, it then follows that for $s = 0, \ldots, n-1$, the function $y_\tau^{[s]} \equiv Y_{s+1,\tau}$ has the asymptotic form as $t \to \infty$,

$$y_\tau^{[s]}(t) = [\lambda_\tau^{\,s} + o(1)]\, q(t)^{-\alpha_{s+1}}\, r(t)^{-\beta_{s+1}} \exp\left[\lambda_\tau \int_a^t (q/r)^{1/n}\right]. \qquad (10)$$

A straightforward computation gives that $D_1 = D_2$ if and only if $m = k = 1$. In this case it is possible to simplify the hypothesis of the Theorem 1. For $D_1 = D_2$, an inspection of the above proof yields that the theorem may be stated with the hypothesis $(q/r)^{1/2} \notin \mathbf{L}(a, \infty)$ and each of $[(q/r)^{-1/2}(r'/r + q'/q)]'$ and $[(q/r)^{-1/2}(r'/r + q'/q)^2] \in \mathbf{L}(a, \infty)$. With these conditions we now derive Theorem 13 of [1]. For $\mu = (qr)^{-1/4}$ and $p = r$ we have that

$$(p\mu^2)^{-1} = (q/r)^{1/2},$$

$$p(\mu')^2 = (q/r)^{-1/2}(r'/r + q'/q)^2/16$$

and

$$p\mu\mu' = (-1/4)(q/r)^{-1/2}(r'/r + q'/q).$$

Hence the conditions $(p\mu^2)^{-1} \notin \mathbf{L}(a, \infty)$ and $\mu(p\mu')' \in \mathbf{L}(a, \infty)$ imply by Lemma 5 of [1, Chap. 4] that $p(\mu')^2 \in \mathbf{L}(a, \infty)$. Since

$$(p\mu\mu')' = \mu(p\mu')' + p(\mu')^2,$$

we also have that $(p\mu\mu')' \in \mathbf{L}(a, \infty)$. By the above remarks, this completes the proof.

In the important special case $r \equiv 1$, the hypothesis of Theorem 1 can be simplified.

COROLLARY *If* $r \equiv 1$ *and* $[q^{-(1+1/n)}q''] \in \mathbf{L}(a, \infty)$, *then the hypothesis of Theorem* 1 *is satisfied.*

Proof. Let $p = q^{(2+1/n)}$ and $\mu = q^{-(1+1/n)}$. Then

$$\mu(p\mu')' = -(1 + 1/n)\, q^{-(1+1/n)}q''$$

so that the hypothesis of Lemma 5 of [1] holds. If

$$p(\mu')^2 = (1 + 1/n)^2\, q^{-(2+1/n)}(q')^2 \notin \mathbf{L}(a, \infty),$$

we have by Lemma 5 that $p\mu' = -(1 + 1/n)\,q' \to \gamma \neq 0$ as $x \to \infty$. Moreover $\gamma > 0$ since $\mu > 0$. Hence by l'Hospital's rule, $q/x \to -n\gamma/(n+1)$ as $x \to \infty$ which implies $q < 0$ for large x. This contradiction yields $p(\mu')^2 \in \mathbf{L}(a, \infty)$. Thus

$$[q^{-1/n}q'/q]' = [q^{-(1+1/n)}q'' - (1 + 1/n)\,q^{-(2+1/n)}(q')^2] \in \mathbf{L}(a, \infty).$$

Applying Lemma 5 once more,

$$p\mu\mu' = -(1 + 1/n)\,q^{-(1+1/n)}q' \to \delta$$

as $x \to \infty$. Thus by l'Hospital's rule, as $x \to \infty$,

$$(q^{-1/n})/x \to \delta/(n+1).$$

If $\delta \neq 0$, $xq^{1/n} \to (n+1)/\delta$ as $x \to \infty$, and if $\delta = 0$, $xq^{1/n} \to \infty$ as $x \to \infty$. In either case we have $q^{1/n} \notin \mathbf{L}(a, \infty)$ and the corollary is proved.

For the differential equation

$$(x^u y^{(m)})^{(k)} \pm x^v y = 0$$

on $[1, \infty)$, Theorem 1 is applicable if $u - v < n$. In this case it is sufficient to take the y_τ so that Eq. (10) reduces to

$$y_\tau^{[s]}(t) = [\lambda_\tau{}^s + o(1)]\, t^{-(v\alpha_{s+1}+u\beta_{s+1})} \exp[\lambda_\tau n t^{(v-u+n)/n}/(v - u + n)].$$

Since the characteristic roots λ_τ are solutions either of $\lambda^n + 1 = 0$ or $\lambda^n - 1 = 0$, all complex roots will occur in complex conjugate pairs. Hence the solutions given by Eq. (10) can be combined to obtain real-valued asymptotic solutions. We now give these formulas for $r \equiv 1$ and $n = 3$.

If q satisfies the conditions of the above corollary and y is a solution of $y''' + qy = 0$ on $[0, \infty)$, then for some constants a_0, a_1 and a_2 we have

$$\begin{aligned} y^{(s)}(t) = q(t)^{(s-1)/3} \Big\{ & a_0[(-1)^s + o(1)] \exp\left[-\int_0^t q^{1/3}\right] \\ & + a_1 \exp\left[(1/2)\int_0^t q^{1/3}\right]\left[\sin\left\{(\sqrt{3}/2)\int_0^t q^{1/3} + s\Pi/3 + a_2\right\} + o(1)\right]\Big\} \end{aligned}$$

for $s = 0, 1, 2$. If y is a solution of $y''' - qy = 0$ on $[0, \infty)$, then for some constants a_0, a_1 and a_2 we have

$$\begin{aligned} y^{(s)}(t) = q(t)^{(s-1)/3} \Big\{ & a_0[1 + o(1)] \exp\left[\int_0^t q^{1/3}\right] \\ & + a_1 \exp\left[(-1/2)\int_0^t q^{1/3}\right] \\ & \times \left[\sin\left\{(\sqrt{3}/2)\int_0^t q^{1/3} + 2s\Pi/3 + a_2\right\} + o(1)\right]\Big\} \end{aligned}$$

for $s = 0, 1, 2$.

References

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